THE
MATHEMATICAL GAZETTE.

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MATHEMATICAL ASSOCIATION TEACHING COMMITTEE.

REPORT ON THE TEACHING OF MECHANICS.

General aims of the subject.

There is perhaps no branch of mathematical instruction for which a pupil comes prepared with a larger body of intuitional knowledge than he does in the case of Mechanics. Every boy knows that his weight is more effective near one end of a see-saw than near the middle; that a heavy door is more easily pushed open the further away from the hinges he puts pressure on it; that it is difficult to make a sharp turn when he is running very fast; that when he kicks a football, not only does his foot act on the ball, but the ball acts on his foot, and he expects and depends on this action, and he knows that if he misses the ball, not only the behaviour of the ball itself, but his own behaviour also, will be different from what he had intended. It is unnecessary here to multiply examples. The suggestions made in this report are based on the view that this body of knowledge should form the foundation of the teaching, and that the aim of the teaching should be largely concerned with the development of a taste for such accurate thought and consideration of mechanical facts as will make them more intelligible, increasing the interest which attaches to the mechanical behaviour of things, and leading to that insight which brings this behaviour more completely under control.

To be taught as part of Mathematics or part of Science?

The question will naturally be asked: "Should mechanics be taught by mathematicians or by the science masters?" Much stress is laid here on giving reality to the subject, and it may be said that if it would take a generation to make the teaching of mechanics sufficiently real in the hands of the mathematician, it should be handed over to the physicist. However, in view of the progress of recent years, the Committee have no hesitation in recommending that it should be taught by the mathematical staff, but that its position should be that of a link

subject, though the methods adopted should approximate more closely to those used by the science master rather than those used in former times by the mathematicians. If the mathematicians hand the subject over to the science staff a great opportunity for correlating the two subjects will be lost, mathematics will lose one of its most vitalising influences, and science will suffer in that mathematical knowledge will not be so ready for application to other subjects.

Position, and relation to other subjects, in the school curriculum.

The opinion of the Committee is that with a free use of scale-drawing and graphs, the development of the subject in its earlier stages does not depend on a previous knowledge of trigonometry, still less of the calculus, but that it is in connection with such a subject as mechanics that these subjects are most profitably studied. They recommend that the teaching of mechanics should be begun early. About thirteen is the age suggested. But at this age there is no need to separate the subject from geometry, with which it is so closely allied. Very elementary geometry would gain by more attention to points, lines, and angles as they are involved in the determination of position, and by attention to these facts in geometry, some of those facts in mechanics which make their appearance as empirical rules, and require so-called experimental verification, appear as the statement of simple obvious truths.

It may be useful here to develop a little the implications of the statement made above, that mechanics should be taught as a "link subject" between mathematics and physics.

First with reference to mathematics. In the Report of the Joint-Committee of the Mathematical Association and the Public Schools Science Masters' Association issued in 1909, it was recommended that certain practical exercises which generally appear in school syllabuses under the head of hydrostatics should be taught as part of the course in mathematics. We endorse that recommendation but proceed to point out that much other matter, usually taught as mechanics, may with advantage be incorporated in the curriculum in "pure" mathematics and taught in lessons on geometry, algebra or trigonometry. Speaking generally it will be both legitimate and convenient to deal thus with all questions concerning motion where no mechanical principles are involved and where the arguments may be presented as an attempt to bring geometrical or algebraic analysis to bear upon matters of everyday experience. This description would, for example, apply to the geometrical study of displacements, of relative motion and to any of the simpler uses of vectors employed to represent movements. It would also apply to the determination, by graphic or algebraic methods, of the distance travelled in a given time by a point moving in accordance with a given law of speed and other topics related to this. Some will consider that it may be taken, further, to include the study of vibratory motion, regarded as the graphic correlative to the sine and cosine functions, and the extension of this study to the simple consideration of progressive wave-motion. None of these subjects involve the properties of any entities beyond space and time, they offer excellent opportunities of bringing interest and "reality" into pure mathematics, and it is a great advantage to the pupil to have studied them before he comes to apply them to the elucidation of questions that are in the proper sense mechanical.

With reference to physics, the point to be emphasised is that it is highly undesirable to make too definite a distinction between the study of mechanics in the narrower sense and the study of problems such as

those of hydrostatics and what is often called "general physics" or "properties of matter." These are all mechanical problems in the sense that their solution depends, as a rule in a fairly direct and simple manner, upon the application of the ideas of force, moment, work, etc. There seems no reason for excluding them from the course except the bad one that they do not offer much scope for elaborate exercises in pure mathematics, and their inclusion—apart from considerations of their intrinsic importance—would do much to prevent the study of mechanical principles from becoming abstract and academic. It need only be added that the reasons which point to the incorporation in the course of a simple study of such subjects as capillarity and Torricelli's theorem also indicate that at a later stage attention should be given, either in the course of mechanics or in close connection therewith, to the simpler applications of the principles of simple vibration in the theory of sound and in the study of such topics as the vibrations of magnets.

Choice of material.

The extent and variety of those intuitive mechanical notions which even our youngest pupils possess, and which it would be the gravest mistake for us to ignore, place us under the obligation of using some principle of discrimination as to those facts of their experience to which in our lessons we should invite their attention. It does not follow that because a particular mechanical fact is simple and familiar, it should be considered early in our mathematical teaching. So are many facts, and we cannot consider all. There are facts which are simple and familiar enough, which give the clue for the explanation of others, and these are the ones which are most profitably taken early, and which serve best to encourage that interest in the subject upon which our success depends.

Thus the principle to be adopted in mathematical teaching is that topics should be chosen to which our pupils are able to apply *accurate* thought quite honestly, and such as will reveal, by the application of such thought, those relations between facts which are the essence of mathematical knowledge. It is the wealth of this fund of intuitional knowledge which a pupil has directly available, only wanting in that precision which mathematical thought can give, which makes the application of this principle in mechanics not only easy and natural, but one which gives immense satisfaction to the pupil.

Order of treatment.

The development of these intuitions of "common sense" into clearly defined mathematical ideas constitutes the history of the theory of mechanics. It is by no means necessary that a pupil should traverse the long and often tortuous path by which a science has reached its modern form, but some knowledge of its evolution is always helpful to the teacher and is sometimes well-nigh indispensable. In the case of mechanics, for instance, it is impossible to appreciate certain important differences of opinion about the best methods of teaching the subject unless the historical origin of those differences is understood. For this reason, among others, it has seemed desirable to supplement this Report by a brief commentary (Appendix A) upon the genesis of the leading ideas of mechanics which may be of service to teachers who are not familiar with such treatises as Mach's *Science of Mechanics*, and the works of later critics.

There were, it may be said, three logical stages in the development of mechanics. Its first task was to distinguish between the different kinds

of phenomena that challenge investigation and to find methods of exact measurement appropriate to each kind. The second was to discover the principles that govern the behaviour of material bodies in the various circumstances thus distinguished. The third was to reduce these principles to the unity of a single logical system. The most important distinction reached in the first stage is the distinction between "static" and "kinetic" phenomena: that is, between the behaviour of bodies at rest or in uniform rectilinear motion and their behaviour when the speed or direction of their motion is changing. To this there corresponds, in the second stage, the distinction between static principles such as the law of the lever, and kinetic principles, such as the principle of the conservation of momentum. In the third stage the theory of mechanics was faced by a dilemma: Was the unification into a single system to be brought about by means of principles of the former or of the latter kind? In other words, Was the clue to the kinetic behaviour of bodies to be sought in their static behaviour, or were static phenomena to be explained by reference to kinetic?

As is shewn in Appendix A, both modes of procedure are possible and both have at different times been adopted. The development of mechanics in the eighteenth century which culminated in the great work of Lagrange followed, on the whole, the former direction: all mechanical phenomena, whether those of equilibrium or those of accelerated motion, were brought under the static principle of "virtual work." But during the nineteenth century—mainly as a result of the concurrent development of physics—there emerged a tendency to reverse the procedure and to base the whole theory of mechanics upon kinetic principles. This change in the direction of the modern movement is the chief cause of the diversity of opinion among teachers as to the order that should be followed in instruction. Those who attach more weight to the principle that the order in which the elements of a science are taught should follow more or less closely the line of evolution of the science in history prefer to give a fairly systematic treatment of static questions before introducing their pupils to kinetic problems. They support their preference by emphasising the greater simplicity of static problems and their readier amenability to experimental treatment. Those, on the other hand, who hold it desirable to give their pupils from the outset the clear view of fundamental principles which is contained in the modern dynamical theory prefer to start with the study of kinetic problems and to interpret static phenomena in the light of the results derived therefrom. These teachers point to the superior interest of phenomena of movement and contend that the alleged difficulty in dealing with them can be avoided by a suitable mode of approach. In addition there is a third body of teachers who seek some kind of compromise between the two extreme opinions.

In the judgment of the Committee each of these points of view is legitimate. Moreover, the Committee recognise that in different circumstances the subject may be most profitably developed in different ways. The study of mechanics in schools has suffered in the past and is still suffering, because some examining bodies refuse to recognise that there are several methods of teaching the subject, and that liberty should be left to each school to develop the subject on its own lines.

Appendix B gives several schemes suggesting different methods of treatment.

Terminology.

Apart from the issue immediately involved in the controversy just considered there is a related question of some importance. Although

the two conceptions of force which are, respectively, the fundamental notions in statics and kinetics can always ultimately be resolved into one, they cannot but appear substantially different to a beginner. For this reason a pupil who receives his first precise notion of force from the study of static problems is often puzzled, if not baffled, when he meets in kinetics with the measurement of force as rate of change of momentum; it is difficult for him to bring into proper relations with one another ideas derived from contexts so different. Similarly, there is in the notion that equilibrium is only a special case of accelerated motion an element of artifice which must generally prove an obstacle, if not a stumbling-block, to the pupil who approaches statics from kinetics. It seems, therefore, worth while to suggest that notions whose identification offers some degree of difficulty should, at least in the earlier stages, receive distinctive names. There is no need to invent a new nomenclature. Reserving the term "force" for the case of accelerated motion, we may use in other cases the readily intelligible terms "thrust" and "tension." No difficulty need be anticipated in the cases where the terms "force" and "thrust" or "tension" must be used side by side, and it would probably be easier than it is with the present practice to generalise the use of the term "force" if and when it was thought desirable to do so.

Practical work, and the importance of giving reality to the subject.

The emphasis placed above on the necessity of making intuitive knowledge the principal basis of the teaching is not in any way intended to depreciate the importance of experiment. It should perhaps be stated at once that experiment is essential to the efficient teaching of the subject. Its rôle, however, is not primarily to provide a source of intuition (which has its real source in common experience) or even to attempt to verify those notions which experience has given. Such notions are too well founded and are too well relied on to need any such verification. Its rôle primarily is to bring actual mechanical phenomena under direct observation in the class-room, and to exhibit them in more simple guise than they are met with under ordinary circumstances, and under conditions suitable for quantitative work. Experiment will be appropriately used to verify the conclusions or the surmises to which our investigations have brought us.

In the past mathematicians have too often, on the one hand, laid excessive stress on mechanics as an application of mathematics, and have tended to undervalue the principles of the subject itself: in other words, instead of basing the study of mechanics on a number of intuitions and experiments they have tended to use it as a basis for mere algebraic manipulation; or, on the other hand, they have gone to the opposite extreme in attempting to build it up on the minimum number of assumptions. This mathematical aspect has its value, but its place is at the end of the subject rather than at the beginning; in some cases it can be undertaken at the top of a school, but in general, it should not find a place in an ordinary school course ending at the age of sixteen.

As opposed to this excessively mathematical treatment of Mechanics, the "Applied Mechanics" of the engineers has grown up; that subject started by ignoring almost entirely its mathematical aspect. It is worthy of note that in the course of the last few years the engineers have made their applied mechanics more mathematical, and the mathematicians have made their work much more real and have postponed the logical building up of the subject. In fact, we may say that the mathematician and the engineer have been approaching one another in their treatment of mechanics and that

now there is reasonable hope that they may agree on a common treatment of the subject so far as it concerns pupils of school age. There is still much to be done in the way of rationalising the treatment of mechanics as given in our schools. Examinations are largely responsible for the slowness of the desired change, but the Committee are glad to record that the examinations of the Civil Service Commission and the qualifying examination for the Mechanical Science Tripos are setting an excellent example in the matter.

Important as it is that the pupils should do practical work in mechanics, it is of still greater importance that the teacher should have practical experience. When this becomes the rule, instead of the exception, many of the hindrances to the teaching of mechanics on lines suitable for school work will automatically disappear.

Experiments should, when possible, be performed by the pupils themselves, and of course lectures should be illustrated by experiment. So far as possible the experiments should be performed with substantial weights and apparatus; these help to bring home the reality of phenomena, and anyone who has taught pupils doing practical work, will have realised how much more easily principles are exhibited when the apparatus is actually set up and can be handled.

Experiments in which a series of observations are made, to be recorded in tables or graphs which suggest a law, are of special value.

The use of a room for mechanics which contains tables instead of sloping desks, so as to be convenient for carrying out simple experiments or for working with drawing boards (which should be always readily available), is really a necessity for the satisfactory teaching of the subject. It would indeed perhaps be well if, for general purposes, the mathematical class room were furnished in this way.

Exercises and examples.

The Committee feel that it is time that artificial questions should be suppressed; there is plenty of material to be found in real problems related to common experience and to practical and engineering work; these are enough for the time available, and provide enough material to develop the logic of the subject.

Questions asking for simple mechanical contrivances should be given a sufficiently important place, requiring the pupil to devise, or to recall the use of some device such as the belt or chain, the lever, the wedge, the screw, struts and ties, the crank, the eccentric, the flywheel, the escapement, devices for fastening doors and shutters of different kinds (folding doors, doors of railway trucks), valves, springs, catches, triggers, traps, clips for particular purposes, and so on.

Clear recognition should be obtained of the fact that the conditions which determine a given phenomenon are usually very complex, and that a mathematical formula is accurately true (and often very useful in practice) provided it correctly states the relations between the facts which are contemplated, even though in practice it may be only possible to realise such conditions approximately. A sense of the necessity and of the value of making simplifying assumptions should be cultivated. It is the fact that we may deal mathematically with the various aspects of a phenomenon one at a time, which explains the power of mathematics to bring us towards a full understanding of it.

Statics: some points of importance.

If the teaching of the subject begins with statics, the triangle or parallelogram or vector law should be introduced as an experimental

fact. Later, when force is defined more fully, it must be made clear that such a law is involved in the definition itself of the measure of force.

Attention may be usefully called to the fact that the statical equilibrium of a body is not in general the case of a mere fortuitous balance of forces. It characteristically depends on the circumstances being such that while the body is acted on by a force of definite value P, it is supported by an opposing force or "reaction" which becomes adjusted to equality with P.

Thus suppose a heavy body is placed on a table, and that it rests in a state of stable equilibrium. The body is acted on by a force of definite value P, equal to its weight; but from the instant of first contact, the table exerts on it a force opposed to P. This opposing force depends on a state of strain (relative displacement of parts) into which the table is more or less thrown, the strain increasing and the dependent force with it, until the latter is sufficient to balance the weight P of the body.

The equilibrium of any portion of the body itself which is supported by the portion of the body beneath it, is similarly seen to depend on a state of strain into which that portion is thrown.

The rigid bodies of mechanics are those in which, in the circumstances in which they are considered, the relative displacements of parts involved in these strains may be regarded as negligible quantities.

The fact that every part of a body is acted on by gravity should be clearly recognised, though for certain cases this action may be represented by a single force at the centre of gravity.

Kinetics: one-dimensional and two-dimensional motion.

In regard to kinetics, the question has been debated as to whether or not at the outset the subject should be restricted to the consideration of rectilinear motion. Those who advocate this course argue that pupils who (while not ignoring the general case) in the first instance limit their consideration of quantitative work to rectilinear motion, get a quicker grasp of the ideas of force, momentum and energy, and readily take the extra steps necessary to give them a thorough grasp of the general case.

On the other hand, those who oppose this procedure use similar arguments to those which are used in condemning the neglect of solid geometry in the early stages of geometrical teaching, and in particular they urge that since forces are in reality by no means confined to the particular direction in which a body may be moving, a study of kinetics restricted to one dimension fails to throw light on the actual mechanical behaviour of things, and often engenders misconceptions as to the nature of acceleration and force, which are afterwards difficult to remove.

The Committee are of opinion that provided the considerations which are involved are clearly recognized, the choice between these two methods of procedure may well be left to be decided by the requirements of the circumstances of particular cases.

When acceleration is dealt with in connection with linear motion, and determined as the gradient of a speed-time graph, care must be taken to guard against the misconception that acceleration can in general be determined by a method of this kind. For while in the case of rectilinear motion the gradient of the speed-time graph represents the whole acceleration, in general it represents merely the resolved part of the acceleration in the direction of motion, the whole acceleration being represented by the gradient of a graph derived from the velocity curve (hodograph), just as the velocity itself is represented by the

gradient of a graph derived from the position curve or actual path traversed.

Rotatory motion.

It must be considered a weakness in the traditional curriculum that it gives so small a place to the study of rotatory motion. The ordinary pupil who spends a fair amount of time in studying mechanics may learn about "motion in a circle"—usually as a kind of lemma to harmonic motion—but rarely reaches the far more simple and important ideas which are locked up in a separate subject called by the forbidding name of "rigid dynamics." It would seem that only one plausible reason can be advanced in defence of this state of things: namely the difficulty of dealing with the computation of moments of inertia. As the practice spreads of teaching the integration of x^n in simple cases even this reason will lose what cogency it possesses. In any case there is no reason why the kinematics of angular motion should not be considered side by side with linear motion, nor why the equation of energy should not be applied to determine the behaviour, under given forces, of wheels, etc., that may be treated as thin hollow cylinders and to bring out the striking analogy between the fundamental kinetic formulae for linear and rotatory motion. With no more than this modest amount of technical preparation the student would be ready to appreciate the behaviour of his bicycle and to understand at least the broad principles underlying the fascinating applications of gyroscopic motion.

Mass and weight.

Some confusion between the notions of mass and weight is inevitable on account of the fact that the weight of a body in a given locality is proportional to its mass, and that the body which is usually recognised as having unit mass is a body of unit weight, so that the numbers which measure mass and weight are identical. Indeed, in popular language the one term weight is used in the two senses. Confusion in the minds of students of mechanics is best avoided not by comparing the meanings of the terms, but by giving precision to them separately, appealing independently to those experiences from which they have already derived intuitive knowledge of each, and using this knowledge in each case as the basis of the precise idea.

This is less difficult than it is usually thought to be, owing to the fact that notions of mass are intuitively possessed, and this fact needs more recognition.

For the purpose of giving this precision to the notion of mass, it is necessary to recall and contrast effects in the cases of more and less massive bodies, in which weight does not play any part—for example, the bat and ball in cricket, charging in football, experiences in stepping from a rowing boat and from a liner, giving a spin to a bicycle wheel and to a massive flywheel. The simplest and most instructive cases are those of collision, suggesting as they do the value of the mass ratio itself.

The mass ratio for two bodies A and B is better defined in terms of the direct ratio of the effect produced by A (in B) to that produced by B (in A), than by the inverse ratio of the effect produced in A (by B) to that produced in B (by A). The former associates the big mass with the big effect.

Its definition in terms of velocity change is quite as fundamental and correct as its definition in terms of acceleration, and is more easily grasped.

Weight units and absolute units.

In passing from the idea of mass to the measure of force as rate of change of momentum, it is very important that the connection between weight units and the corresponding absolute units should be made perfectly clear, so that it is possible to pass from the one to the other with ease. If in his studies the student is familiar only with absolute units, he will lose by not being able to appreciate the magnitudes of forces in terms of the naturally familiar weight units. It is quite permissible in the more elementary parts of mechanics and physics to use weight units exclusively, though in the more advanced work absolute units are almost indispensable.

APPENDIX A.

Notes on the development of mechanics.

A person who is lifting or supporting a heavy body, or is bending a beam or stretching a cord, or is moving himself rapidly or causing another body to move or cease to move, to alter its speed or to change its direction of movement, has always (if he attends to the matter) a consciousness of effort. This experience of effort is the psychological origin of the notion of " force " (*vis*) which, in one or other of the forms it has assumed, has always been and remains the master-idea of mechanics. There is a tendency, universal in humanity, to " eject " into physical bodies something of the feelings which we should ourselves experience in situations that can be regarded as analogous to theirs. This tendency asserts itself when, for example, we contemplate the " soaring pinnacle " or impute stubborn steadfastness to the Norman tower that stands " four-square to every wind that blows." (In German the process is described by the expressive term *Einfühlung*.) It was, then, natural that men should attribute force or *vis* to the weight that " seeks " the ground, to the string that " prevents " it from falling, to the column that " holds up " the roof, to the " flying " projectile, to the gunpowder gases that " propel " it, to the magnet that " attracts " the iron, and, in short, to dead matter throughout the range of its physical and chemical behaviour. But the mere attribution of *vis* carries us only a little way towards the interpretation of mechanical phenomena. Substantial progress could be made only in so far as the various aspects of the mechanical behaviour of bodies became clearly distinguished from one another, determined by measurement and resumed in formulae.

A characteristic episode in this process was the famous controversy between the schools of Descartes and Leibniz as to whether the " force " of a moving body should be measured by the product mv or the product mv^2. As is well known, the dispute was closed only in 1742, when D'Alembert pointed out that the special function of the former product is to determine the *time*, and that of the second to determine the *distance*, in which the moving body would be brought to rest by a constant resistance. Thus, as the result of prolonged argumentation, the original hazy notion of the *vis* of a moving body became replaced, for scientific purposes, by two precise mathematical concepts: the one denoted by the Galilean term " momentum " (or the Newtonian " quantity of motion "), the other by Leibniz's term " *vis viva*." Meanwhile Newton (following and completing the ideas of Galileo) had reduced to a similarly precise form another intuition of " force "—namely the *vis impressa* that is manifested in the changes of speed and direction of moving bodies—taking as its measure the rate of change of momentum, or, as he himself called it, the *mutatio motus*. It is instructive to note

that down to the middle of the nineteenth century the concepts defined by the products mf and $\frac{1}{2}mv^2$ continued both to be described by terms involving the word *vis* or its equivalents in modern languages. It was only in 1852 that Lord Kelvin (adopting a suggestion of Thomas Young) introduced the use of "kinetic energy" as a substitute for the old "*semi-vis viva*" and so removed a long-established obstacle to clearness of thought. And even to this day the term "principle of *vis viva*" (*forces vives, lebendige Kraft*) remains in our text-books as a relic of a bygone phase of the struggle of mechanical science towards precise and coherent expression.

Long before the age of Galileo and Newton men had seen how to apply a standard method of measurement to another class of manifestations of *vis*—those which we call "static" forces. It was perceived that the action of a supporting beam, of a lever, of a pulled cord, etc., could always be replaced, actually or hypothetically, by the equivalent action of a weight of appropriate magnitude, suspended by means of a suitably attached and directed string. Thus it was easy to estimate all these numerous exhibitions of *vis* in terms of one of them: namely, the *vis* of a suspended weight.

As an illustration of this method and of the theoretical developments which it made possible, the following simple example may be given. Imagine a square frame, $ABCD$, composed of loosely jointed, weightless bars, hung by a smooth peg at A. Let a weight W be suspended at C and let the frame be prevented from collapsing by the insertion of a (weightless) rod BD. Then the first question that arises is: In what numerical terms may the *vis* exerted by the rod be defined and measured? To answer this we may suppose the action of the rod BD to be replaced by that of two equal weights attached to B and D by light cords which pass horizontally over smooth pegs. Suppose it to be found that, to preserve the square shape of the frame, the weights must be each of P pounds; then we may, conveniently and with obvious propriety, describe the thrust exerted by the rod as a thrust of P pounds-weight.

But there is a further question involving much wider considerations: Is it possible to *calculate* (*i.e.* predict without actual measurement) the value of P in a given case? To deal with this inquiry we may (for convenience) suppose the strings to be of such length that the centres of gravity of W and of the two weights P lie in a horizontal line. Contemplating them in that position we may argue as follows. This system of bars and weights is perfectly free to move under the influence of gravity within the limits which its structure leaves open to it. If no movement occurs it must be because the joint centre of gravity of the three weights is already as low as, in the circumstances, it can be. For if, by any movement now possible, the centre of gravity could be lowered further, experience of the ways of heavy bodies has shewn that that movement would certainly take place. A disturbance imposed by pulling C slightly up or down would leave the joint centre of gravity momentarily at its present level and would then cause it to rise. Let us suppose that such a disturbance caused W to rise or fall a very small distance d; then elementary geometrical considerations shew that the weights P would fall or rise a distance $\frac{1}{2}d$. It follows, if the centre of gravity of the system is momentarily to remain at its present level, that $P=W$. That is, we may conclude that in the situation originally in view the thrust in the rod BD was equal to the weight suspended at C.

It will be noted that this theoretical problem has been solved by the application of a principle derived from intuition of the behaviour of bodies free to move under gravity. In the form here used this principle appears to have been first clearly grasped by Huygens: later it was

elaborated into the technical method of "virtual velocities" or "virtual work." It is important next to observe how the simple ideas upon which the use of this method is based were carried over, by the great mechanists of the eighteenth century, from the field of "static" to the field of "kinetic" forces.

The first thing needed is to establish a principle of equivalence between the measures of the two kinds of forces. That is an easy matter. It may be taken as obvious that the *vis* of a weight when suspended and its *vis* at the moment when, the string having been cut, it begins to fall, are identical, being both expressions or correlatives of the same action of the earth. Thus a weight which, when measured statically, has the numerical value M, has, when measured kinetically, the numerical value Mg. Now suppose a particle, of mass m, to be moving above the earth's surface with acceleration f—that is, under a kinetic force or *vis impressa* mf. This particle may (at least hypothetically) be made to continue its journey in equilibrium—*i.e.* to continue in its present direction but at a *constant* speed—by attaching to it a string passing in a suitable direction over a smooth peg and bearing a weight of suitable magnitude M. If, as the particle moves on, the impressed force changes, correlative changes must, of course, be made in the adjustment of the string and the magnitude of the suspended weight; but at any moment the connection between M and m will, by the preceding argument, be given by the equation $M = mf/g$.

Let us advance from this simple case to the consideration of a system consisting of a swarm of particles, shielded from gravitational influence, but moving freely, each under the action of all the others. Under these mutual actions let the swarm pass during a very short interval from one configuration C_1 to another C'_1. Then, by the foregoing argument, it would hypothetically have been possible, by a suitable adjustment to each particle of strings and weights descending into the gravitational field, to make the swarm move from configuration C_1 to configuration C'_1 in equilibrium. During the passage some of the weights would have gone up, some down, and a certain total of "virtual work" would have been done which may be called w_1. Let us next suppose the passage in equilibrium from C_1 to C'_1 to be repeated with the difference that the various actions of each particle upon the others are all to be suppressed and replaced by a second system of strings and suspended weights. In this case there will be, in addition to w_1, a second amount, w_2, of virtual work performed by the second group of weights. But we have now a situation precisely comparable with that contemplated in the case of the loaded frame. We may, therefore, apply to it the same principle with regard to the position of the centre of gravity of the whole body of weights, and assert that

$$w_1 + w_2 = 0.$$

Repeating the argument for each of the minute displacements into which the transition from the original configuration C_1 to any other configuration C_2 may be regarded as broken up, we finally conclude that

$$W_1 + W_2 = 0,$$

where W_1 stands for the total amount of virtual work of the weights used to replace the impressed forces, W_2 the corresponding total for the weights introduced as substitutes for the mutual actions of the particles.

It is unnecessary to trace the steps by which Lagrange translated this abstract generalisation into concrete results of the highest importance. It is, for our purpose, sufficient to have illustrated, in principle, the way in which general problems of motion may be, and as a matter of historical fact have been, reduced to problems of equi-

librium—in other words, the way in which kinetic questions may be and have been translated into static questions to be solved by an appeal to what may be called static intuitions.

Some discussion of the matter seemed necessary in view of its close bearing upon fundamental questions of procedure in teaching. Since the publication of Helmholtz' classic memoir (1847) and Thomson and Tait's *Natural Philosophy* (1867), there has been a tendency for mathematicians, while accepting and developing Lagrange's technical methods, to seek the basis of their validity in Newton's third law and in the physical principle of the conservation of energy (of which the third law is a partial expression) rather than in Huygens' statical principle of the centre of gravity. In other words, there is a tendency to examine questions of equilibrium in the light of general ideas derived from the study of accelerated motion rather than to follow the older method of procedure.

T. P. Nunn.

APPENDIX B.

SUGGESTED COURSES IN ELEMENTARY MECHANICS.

Scheme No. 1. By A. W. Siddons.

A Use of Spring Balance.

B The law of the lever.
Moment of a force.
Machines, work, efficiency.
Horse-power.
Centre of gravity.

C The triangle of forces.
Applications of the triangle of forces.
Three-force problems.
Polygon of forces.
Friction.

D Resolution.
Friction.
Centre of gravity.
Frames.
General equilibrium.

Motion in a straight line.

E Space-time graphs ; Velocity-time graphs.
Average velocity when velocity is increasing (or decreasing) uniformly.
Falling bodies.
Impulse—momentum.
Work—energy.
Force and acceleration.

Motion in two dimensions.

F Relative velocity.
Projectiles.
Inclined forces.
Motion in a circle.
Simple harmonic motion.

G Rotating bodies.

The above course is designed for beginners in mechanics ; it is intended that it should be based on intuition from general experience and on direct experiments.

Intuition should be appealed to whenever possible ; the object of experiment is (i) to bring out the quantitative aspect of topics which intuition has only dealt with qualitatively, (ii) to consider in their barest essentials phenomena which in actual everyday experience are so wrapped round with non-essentials that it is hard to realise what is relevant and what irrelevant.

The two parts labelled B and C are independent of one another, so that they may be taken in alternate terms, and boys may begin with either part. [This arrangement is found very convenient in schools where removes are made every term.] Experience shows that there is very little reason for preferring to take B before C or C before B ; on the whole, perhaps it is slightly better to take B before C.

E is taken immediately after B and C. At first very little stress is laid on the idea of acceleration. The main ideas are :

(i) The *impulse* of a force acting for a given time is measured by the product of the measures of the force and the time.

(ii) The *work* done by a force acting through a given distance is measured by the product of the measures of the force and the distance.

(iii) The *momentum* of a moving body is defined to be equal to the impulse it can exert before coming to rest.

From this the formula for momentum can be found, and it follows that the change of momentum of a body is equal to the impulse exerted on it.

(iv) The *kinetic energy* of a moving body is defined to be equal to the work it can do before coming to rest.

From this the formula for kinetic energy can be found, and it follows that the change of kinetic energy of a body is equal to the work done on it.

After E, topics from D and F can be chosen in various orders.

All this can be followed, if time permits, by a consideration of the interrelation of the various assumptions (from intuition and experiment) that have been the basis on which the subject has been built, but this follows the course and is not considered prematurely.

In the above course the units of force used are entirely gravitational. This is open to the objection that the unit of force is variable, but the variation is slight, so long as our consideration is limited to bodies near the earth's surface. All our initial ideas of force are based on our muscular sensations, and are closely associated with the idea of weight, and the notions thus acquired are sufficient for the course sketched above. If the idea of the absolute unit is taken at all in such a school course, it can come at the end.

As regards the teaching of physics, the absolute unit is not wanted in heat (the mechanical equivalent of heat is done in terms of ft.-lbs.) ; in many school courses it is not wanted in magnetism and electricity, and in those cases in which it is wanted it is easily introduced then and its value is made clear.

A. W. SIDDONS.

SCHEME NO. 2. BY W. J. DOBBS.

In the large majority of Secondary Schools there is a considerable influx of scholarship boys from the Elementary Schools at the age of twelve. The work of such schools may be conveniently arranged in four stages, namely :

Stage I,	- -	age about 10 to 12.
Stage II,	- -	„ 12 to 14.
Stage III,	- -	„ 14 to 16.
Stage IV,	- -	„ 16 to 18.

Under present conditions the vast majority of boys leave school before the end of Stage III.

It is suggested that Mechanics, as presented in this scheme, should enter largely into the mathematical work of Stages II, III and IV, being commenced informally in Stage II and treated practically, but not as a separate subject.

In Stage II, Mechanics may very properly and conveniently start with the topics: Hooke's Law, Young's Modulus of Elasticity, Equilibrium of Three Forces, Principle of the Lever, Principle of Archimedes, the work commencing with experiment and continuing with reference to actual apparatus. It is easy to design practical examples with very simple apparatus in these subjects.

The pupil associates *force* at the outset with *weight* or *earth-pull.* He regards a suspended body as kept in equilibrium by two forces which are equal and opposite and act upon the body in the same line, namely, the earth-pull on the body and the supporting force. The action of the body on its support is the reaction corresponding to the supporting force. Thus it is seen that in addition to the usual notions of length and angle, the boy adds also this conception of force. By this method *statics* may precede *kinetics*, and *kinetics* may be commenced by considering motion in a straight line, as of a train on a straight railway, a shot in the bore of a gun, the piston of a steam engine, or water in a straight uniform pipe. "As we live in a field of force due to the attraction of the earth, it is convenient to take the attraction of the earth on a pound weight as the unit of force, and to call this the force of a pound; this is the gravitation unit of force employed in all practical problems of architecture, engineering, mechanics and artillery; it is always ready at hand, and can be measured with the same accuracy as the operation of weighing."

In Stage III, the Mechanics course gradually assumes a somewhat more formal character adapted to the capacities of the pupils. Mathematical masters, however, are in too many instances apt to make it too formal. It is most important that the examples should be real and not academic. Valuable time must not be "spent in the discussion of neat mathematical unrealities—in the calculation of the behaviour of impossible bodies under impossible conditions." The graphical treatment of velocity and acceleration (in reality *graphical calculus*) may be included, and will lead naturally to first notions of the calculus itself. This work should not be confined to *Uniform* Velocity and *Uniform* Acceleration. Numerous examples in drawing and interpreting distance-time graphs and velocity-time graphs tend to do away with the prevailing notion that velocity when not uniform is uniformly accelerated. The resolution of forces, velocities and accelerations will be included, Elementary Trigonometry and Common Logarithms being used freely. The gravitation measure of force should be continued until there is adequate motive in the work of the school for introducing absolute measures of force, and this may quite properly not occur until Stage IV is reached. Such motive may be found in the consideration of cosmopolitan questions of astronomy and electricity. "The C.G.S. units are extremely small and unsuitable for a practical question, while in Astronomy, the numbers run so large that all human units appear equally insignificant." (It is perhaps worth while to consider whether the M.K.S. system should be introduced into school work.)

In commencing the subject *kinetics*, the following treatment is recommended.

When a body of weight W moves from rest under the action of constant forces equivalent to a resultant F, it does so with acceleration a (in the same direction as F), such that

$$F/W = a/g. \qquad \text{(i)}$$

If, in time t, the body is moved from rest through distance s, acquiring velocity v,

$$F \cdot t = W \cdot v/g. \qquad \text{(ii)}$$

$$F \cdot s = W \cdot v^2/2g. \qquad \text{(iii)}$$

In (i) both sides of the equation are purely numerical. In (ii) both sides have the dimensions force × time. In (iii) both sides have the dimensions force × length. It is convenient to insert the units thus :

$$Ft\,[\text{lb. sec.}] = W\frac{v}{32\cdot2}\left[\text{lb.}\frac{\text{ft./sec.}}{\text{ft./sec.}^2}\right] = W\frac{v}{32\cdot2}\,[\text{lb. sec.}],$$

$$Fs\,[\text{kgm. mtr.}] = W\frac{v^2}{2\times9\cdot81}\left[\text{kgm.}\,\frac{\text{mtr.}^2/\text{sec.}^2}{\text{mtr./sec.}^2}\right] = W\frac{v^2}{19\cdot6}\,[\text{kgm. mtr.}],$$

or even $$Fs\,[\text{oz. cm.}] = W\frac{v^2}{2\times981}\left[\text{oz.}\,\frac{\text{cm.}^2/\text{sec.}^2}{\text{cm./sec.}^2}\right] = W\frac{v^2}{1962}\,[\text{oz. cm.}].$$

Work, including simple Work Diagrams, Machines (Force Ratio, Velocity Ratio, Mechanical Efficiency) and Kinetic Energy $\left(W\times\frac{v^2}{2g}\right)$ should be included, and perhaps the Elementary Statics of simple roof trusses and bridges.

In Stage IV, it is desirable that the absolute measure of force should be introduced, if this has not already been done. The subject matter may include Centre of Gravity, Work and Energy, Momentum, Friction, Projectiles, Simple Harmonic Motion, the Mechanics of Rotation, some Astronomy, and examples requiring the use of Infinitesimal Calculus. A simple treatment of the Mechanics of self-propelled wheeled vehicles may also be included with great advantage. W. J. DOBBS.

SCHEME No. 3. BY G. GOODWILL.

The following syllabus is intended to indicate a scheme for the study of Mechanics by the consideration of mechanical behaviour, as it is exhibited in ordinary familiar experience. In the opinion of the author of the scheme, such a presentment requires that more consideration should be given to kinetics than to statics.

Kinetics has usually been held to be too difficult for young pupils. It is believed that the difficulty has been due to the too early introduction of acceleration. Change of velocity is here taken as the basic phenomenon which characterises mechanical action. Some notion of this quantity is assumed to be intuitively possessed, and quite naturally to engage the interest of the pupil, and there need be little tedium in that part of the work which is devoted to giving it precision. The time and care which the teacher devotes to this and the immediately preceding work on the geometry of position, will be well repaid by the insight which is given to a wide range of mechanical phenomena, as well as by the immediate interest which is evoked. The intention in this early work is to make direct appeal to intuitions regarding directed quantities in space, and to use scale drawing as the means of representing such quantities.

Throughout the course the order followed has been determined by considering what might be a natural order of development in the pupil's mind.

Mechanics is regarded in the scheme as a branch of mathematics, a principal aim being the cultivation of a faculty for the precise formulation of facts and their relations—an aim which it shares with mathematical education in general.

With regard to material arrangements for the course, it is very desirable that a room should be used provided with steady tables and stools rather than desks, and that drawing boards and T-squares should always be readily available.

The special apparatus required is very little. For a form of 24 pupils the following is suggested: 24 spring balances, 6 metre rules (cm. and

in.), 12 stiff laths about 2′ 6″ long, 6 sets of weights (1 to 500 gm.); 12 triangular wooden prisms (to use as fulcra), 6 vector balances and one extended vector balance (Messrs. G. Cussons, Manchester); 1 Fletcher's trolley; 1 heavy wheel and axle with ribbon recorder; 1 stop watch.

Many experiments would be worked by sets of four boys (or more), the experiment being conducted by each boy in the set in turn, himself taking charge and the others assisting him while his record was made.

In the case of experiments where ordinarily only one set of apparatus is available (such as with the trolley), boys may come up in groups of four and make their records and take them away, work associated with the general disposition, tabulation and graphing of their records occupying boys while not actually doing the experiment.

When necessary a large number of duplicates of a graphic record may often be rapidly made by placing it on a number of similar sheets and pricking through. In certain cases such duplicated records are extremely useful.

Subject Matter.	Topics for discussion and exercise, and experimental work.
I. *Position, displacement and velocity.*	
Determination of position in terms of distance and angle. Maps and diagrams. When the position of a place is determined relative to any one place shown in a map, its position may be marked in the map, which will then show its position relative to all the other places shown in the map.	Relative positions of well-known public buildings and towns. Crane problems. Heights and distances. (Scale drawing.)
Resolution.	
Change of position or displacement. Displacements of different bodies may be combined and shown in maps just as positions are; thus the relative displacements of three bodies A, B and C are represented in a displacement map by three points a, b and c; ab representing the displacement of B relative to A, ac that of C relative to A, and bc that of C relative to B.	Buoys, boats and ships, bicycles and trains, the wind, etc. Construction of displacement and velocity maps for solution of problems. (Scale drawing.)
Velocity regarded as constant.	
Velocity maps.	
Varying velocity. Measurement of a change of velocity based on a comparison of the velocities of two bodies whose velocities are different. (Fundamental to the validity of the laws of vector combination in mechanics.) The position curve or path and the velocity curve (hodograph).	Motion of the projectile, falling bodies, the pendulum, bicycles, trains, the trolley, with construction of the corresponding position and velocity curves, the position curves being preferably obtained in the first exercises in each case from an actual record of the motion.

Subject Matter.	Topics for discussion and exercise, and experimental work.
Speed. The distance-time graph and its slope. Speed at an instant.	Distance-time graphs corresponding to the above. (Scale drawing and graphs.)
Position, displacement and velocity as vectors. Vector sums and differences.	
II. *Mass, Momentum and Force.*	
The laws of interaction, viz. Change of velocity of a body is due to action upon it by another body. The changes of velocity produced by two bodies in one another by their mutual action are in opposite directions and in a constant ratio. This ratio is the mass ratio of the bodies by which the changes are produced.	Experience and observation in connection with running, jumping, sliding, cricket, bat and ball, charging in football, guns and projectiles, starting and stopping of trains. Experience in train corridor and standing in bus when it starts, stops or turns a corner.
Determination of mass. Momentum as a vector. Impulse. The conservation of momentum.	The vector balance. Solution of examples connected with the preceding work in this section by graphic numerical and algebraic methods.
Force and its measurement as rate of change of momentum. Weight, gravity, and friction. Absolute units and weight units. Reactions associated with strains. Forces in equilibrium. Tension and pressure ; stress.	Projectiles, falling bodies. Statical experiments with spring balances and weights ; pith balls electroscope ; inclined plane ; trolley ; simple experiments in hydrostatics. Solution of examples connected with the preceding work in this section.
Centre of mass. Consideration of the laws of interaction in connection with the suggestion that the velocity of an extended body or system may be taken to be represented by the velocity of its centre of mass. Comparison of the motion of the centre of mass of a system with that of a particle in similar circumstances. Observation of the fact that it is the point round which a body rotates when free. Definition of the centre of mass of a system of particles as a point so situated that the sum of the mass vectors of the particles from the point is zero.	Appearance of bursting rocket ; shrapnel shells ; experiment with card projected while rotating in vertical plane, and with two balls attached by thread and similarly projected ; experiments with two balls attached by stiff wire, hung up by inverted Y suspension, which is attached to ends of wire, the system allowed to spin round and the centre of mass determined by adjusting a light bead which slides on the wire ; similar experiments with three balls attached to ring of wire ; experiments with a heavy bar suspended at its centre so that it hangs with its length horizontal, and the bar struck by a suspended ball, and observation of the fact that the second law of interaction is fulfilled. (Extended vector balance.)

Subject Matter.	**Topics for discussion and exercise, and experimental work.**
III. *Force as a localised Vector.*	
Moment of momentum and moment of force. A force is completely determined by its magnitude, direction and position.	Further experiments with extended vector balance. The dependence of angular velocity jointly on the magnitude of blow and arm.
Angular velocity.	
Consideration of moments in statical cases. Resultant. Parallel forces. Centre of gravity. The suggestion that it coincides with the centre of mass. Machines and work. The conditions of equilibrium.	Statical bench experiments with spring balances attached by strings to slab of wood, and with stiff lath loaded with weights balanced on fulcrum, and with cards or sheets of metal, determining centres of gravity and verifying conditions of equilibrium.
	Consideration of different examples in statics and solution of problems and exercises.
IV. *The Conservation of Moment of Momentum.*	
The sum of the mass vectors of a system of particles from any point is equal to the mass vector of the total mass, regarded as collected at the centre of mass, from the same point.	(Much of the mechanical experience and experimental work with which this part of the course is concerned has already been introduced and discussed in the preceding work. At this stage the pupil is chiefly occupied with the precise mathematical formulation and the establishment of those principles which have already been suggested and accepted provisionally.)
The sum of the momenta of the particles of a body is equal to the velocity of the centre of mass, multiplied by the total mass.	
The velocity of the centre of mass of a system is not affected by any action on one another of the bodies within the system.	
The moment theorem.	
The moment of momentum of a system and the conservation of moment of momentum.	
If a body is travelling without rotation, the moment of momentum of the body, about its centre of mass, is zero. The moment of momentum of a body about its centre of mass depends only on its rotation.	
The centre of gravity of a body coincides with its centre of mass.	

Subject Matter.	Topics for discussion and exercise, and experimental work.
The angular velocity produced in a bar by a blow is proportional to the moment of the blow about the centre of mass of the bar.	Experiments with simple extended vector balance and same loaded with sliding masses. Experience in attempting to give a sharp twist to a long bar (map pole) round an axis at right angles to its length, when grasping it at its centre, and comparison with giving a similar twist to a compact mass.
Moment of inertia. The parallelism of its relation to rotation (in certain important cases) with the relation of simple mass to translation.	
Rotation of a body round a fixed axis.	Experiments with heavy wheel and axle.
The conditions of equilibrium.	Various statical examples and exercises.
Acceleration. The use of the velocity curve (hodograph) to show the parallelism of the relation of acceleration to velocity with the relation of velocity to position. Acceleration at an instant. Circular motion with constant speed. Normal acceleration. Tangential acceleration, or acceleration of speed. The relations between speed time and distance are the same as those between length, breadth and area. The speed-time graph, its slope and area. The equations of linear motion. Projectiles. Work and kinetic energy. Tangential force the space rate of change of kinetic energy. The relations between distance, force and work are the same as those between length, breadth and area. Work diagrams.	The projectile; the pendulum; vibration of loaded spiral spring; Attwood's machine; cylinder rolling down inclined plane; determination of g; coefficient of elasticity, and the determination of its value for material of balls of vector balance. Various kinetical examples and exercises.
The conservation and dissipation of energy. Simple vibration. The pendulum. Central forces and gravitation.	

G. Goodwill.

Scheme No. 4. By T. P. Nunn.*

The term "mechanics" can be understood in a narrower and in a wider sense. In the following syllabus it is taken broadly, as meaning an investigation of the simpler and more fundamental types of phenomena explicable by means of the concept of "force." Interpreted thus, mechanics as a school subject overlaps physics on one side and pure mathematics on the other. It overlaps physics since many of the typical phenomena explicable in terms of "force"—for instance,

* This syllabus is mainly an expansion of one already printed in the *B. A. Report on the Teaching of Science in Secondary Schools*, 1917.

the buoyancy of fluids, the formation of bubbles, the work done by expanding gases, the modes of vibration of strings, the oscillations of a magnetic needle—are, in the ordinary sense of the term, physical phenomena and are studied in connexion with physical theories. And it overlaps pure mathematics inasmuch as it deals with certain typical forms of motion whose descriptive properties can be determined by means of geometry and analysis without recourse to the idea of force. It is evident, also, that it overlaps astronomy; for many of the most interesting and important phenomena involving force lie within the territory of that science.

The work is set out as a four years' course, to be taken as part of a systematic scheme of instruction in science and mathematics for all pupils between the ages of twelve and sixteen.

First Year.

The lessons of the First Year fall into two groups. In the first group the use of the balance as an instrument for measuring weight leads to an investigation of the principles involved in its action and so to preliminary notions about moments, equilibrium and centres of gravity. In these lessons weight is taken as an ultimate property of bodies which does not call for analysis or explanation. The second group is devoted to a simple study of familiar pieces of mechanism involving levers, cog-wheels, etc. The object of the lessons in this group is to encourage an interest in mechanism and to develop the power of kinematic analysis.

The study of time-measure (B. 1) is best taken in connexion with simple observations on the daily motion of the sun. Such observations serve not only to explain the basis of civil time-measurement, but also to lay a foundation for important kinematic ideas to be developed later in the course of mechanics.

A. *Weight. The balance. Moments.*

1. Density and specific gravity. Determination of weights by the balance and of volume by calculation or displacement.

2. The mechanism of the balance and the conditions for true weighing. The laws of the lever. The grocer's scales. Weighing-machines.

3. The pressure on the fulcrum of a loaded lever. The centre of gravity of a body as a fulcrum, and as the "centre" of the weights of its parts. Experiments, toys, etc., illustrating stable and unstable equilibrium.

4. Simple calculations and laboratory experiments on centre of gravity, etc.

B. *Simple Mechanisms.*

1. Essentials of a mechanism of a simple clock driven by a weight or a spring and controlled by a pendulum. (A single-handed clock, like that of Westminster Abbey, is most suitable.)

2. Isochronism of the pendulum. Effects of loading or changing lengths of pendulum. The "simple" pendulum; connexion with swing-period and length. Experimental determination of simple pendulum equivalent to a given pendulum. The balance-wheel in watches and clocks.

3. Ancient time-measurers: the water-clock, the hour-glass, etc. Graphic records of the amount of water or sand delivered in a given time; uniform, variable and average speed of flow.

4. Examination of common pieces of mechanism, such as a door-lock, the "three-speed" gear of a bicycle, Archimedean drill, egg-beater,

laundry-mangle, etc. (There is scope here for individual work, involving written descriptions aided by diagrams, etc.)

Second Year.

The lessons fall into three groups. Group A continues the work begun in Group A of the First Year. Group B deals with the mechanics of fluids. The lessons of Group C follow up those of Group B of the First Year by more abstract studies of motion, intended to be introductory to the study of kinetics which is to be begun in the Third Year. These lessons are best taken as an integral part of the course in Geometry. The subject-matter of B. 6, 7, may profitably be connected with the study of plant-physiology.

A. *Stress. Strain. Work.*

1. Use of spring balance to measure a tension or a thrust (*i.e.* a "push" or a "pull") in terms of weight. Measure of pressure in terms of weight per unit area. The reciprocal action involved in stress.

2. Strain as the invariable accompaniment of stress. Measurement of longitudinal strain, shearing strain and compressibility (volume strain), in simple cases.

3. Hooke's Law. Simple experimental tests in case of stretched cords and wires, bent beams and twisted wires and rods. Elasticity as the ratio of stress to strain.

4. Practical applications. How strength of materials (*e.g.* steel) is measured. Elastic limits. The "factor of safety."

5. Pulleys. Use of a single (rough) pulley. Measurement of efficiency as the ratio L/P, where L is the load lifted and P the pull exerted.

6. Use of movable pulleys—*e.g.* in raising the jib of a crane, and in the block and tackle. Measurement of efficiency found to require account to be taken of distances (d_2 and d_1) moved by load and power. The formula $E = Ld_2/Pd_1$. The mechanical advantage (L/P) and the velocity ratio (d_1, d_2).

7. The products Pd_1 and Ld_2 regarded as measuring *work*. Definition of efficiency of a machine as ratio of work done *by* to work done *upon* it. Ideal equivalence of the amounts of work when there is no loss by friction, etc. Equality in this case of mechanical advantage and velocity ratio.

8. Loss of work by friction. Simple investigation of friction. Coefficients of friction.

9. Re-examination of levers (First Year, A) from point of view of work. The ideal equivalence of § 7 generalised into the Principle of Virtual Work. Simple illustrations and examples of the use of the principle in determining thrusts and tensions, *e.g.* haulage on smooth inclined plane.

10. Revision of theory of moments (First Year, A). A horizontal beam is loaded at given points with given weights: calculation of tensions in vertical spring-balances used to support its ends. Verification by measurements.

11. Calculations of point where the whole load may be collected so as to produce same tensions in supporting balances. Principle of equivalence of a single thrust or tension to a set of parallel thrusts or tensions. Determination of centre of gravity by the method of moments.

12. The beam of § 10 is loaded at mid-point. One of the supporting balances (R) is made to slope from the vertical. The other supporting balance (P) is kept vertical by attaching a third balance (Q) horizontally at the same end of the beam. By this means the pull in R is "resolved"

into vertical and horizontal "components" in P and Q. The relation between the amounts and directions of the pulls in P and Q and the single pull in R; the Vector Law. (*N.B.* This exercise is to be taken subsequently to C. 1.)

13. Further study of Vector Law. Simple practical applications: the suspension bridge, the crane, cantilever frames, etc.

B. *The Mechanics of Fluids.*

1. Buoyancy. How ships float. Measurement of extra displacement produced by adding "cargo" to a box floating in water suggests Archimedes' Principle. Confirmation in case of other liquids. Extension of principle to bodies that sink. Use of camels and pontoons. Submarine boats. Balloons and airships; contrast with aeroplane.

Exercises on use of Archimedes' Principle in determining volumes and specific gravities.

2. The barometer as a meteorological instrument. Construction of siphon barometer. Pascal's theory of action illustrated by demonstrating increasing pressure at lower depths in a jar of water. The experiment of the Puy de Dome. Reduction of barometer readings to sea-level for construction of barometric charts. Relation between isobars and winds.

3. Boyle's experiments in confirmation of Pascal; leading to notion of the "spring" of the air and to Boyle's Law.

4. Experiments and apparatus illustrating air-pressure; pumps, vacuum brake, parcel-transmitter, siphon, etc. The aneroid barometer: its use in determining heights in mountaineering, aeroplaning, etc.

5. Archimedes' Principle explained by theory of liquid-pressure. The theory applied to explain water-supply systems, hydraulic lifts and engines.

6. Capillarity. Measurement of surface-tension (in grams-weight per cm.) by rise of water in tube. The formula $T=\frac{1}{2}rhd$.

Pressure inside a soap-bubble. The formula $P=2T/r$. Applications to pressure in boilers.

Simple experiments on bubbles, drops and jets. (See Worthington's *Splash of a Drop* and Boys' *Soap Bubbles*.)

7. Osmosis. Passage of dissolved salts through a porous partition until equality of concentration is set up. Practical applications, *e.g.* in purification of beet-molasses. Semi-permeable membranes; law of osmotic pressure; comparison with Boyle's Law for gases. Application to plant-cell.

C. *Graphic Study of Motion.*

1. Exercises in the use of vectors to represent (*a*) relative position, (*b*) relative displacement, (*c*) relative velocity.

2. Other graphic exercises, *e.g.* the cycloid as the trace of a point on the circumference of a rolling wheel; curves of pursuit, etc.

Third Year.

There are three groups of lessons. The first is devoted to a study of the elementary grammar of motion, applied to define and elucidate certain important types of movement in nature. The central problem of the second group is to determine what general conditions or "laws" hold good when one body's motion is determined or influenced by the motion of another. The lessons of these groups are to be taught in close relations with one another. The purpose of the third group is to develop further the inquiries and methods of the Second Year work.

A. *The Analysis of Motion.*

1. Motion may be *one-dimensional* (*i.e.* in a straight line), *two-dimensional* (*e.g.* a boat, a projectile, a point on a revolving wheel, the tip of a pendulum), or *three-dimensional* (*e.g.* a bird in flight, a man climbing a hill or a spiral staircase). The velocity of a moving particle may be either *uniform* or *variable* in respect of (*a*) speed, (*b*) direction, or (*c*) both speed and direction. "Uniform velocity" implies constancy both of speed and of direction; velocity may be "variable" even though speed is constant.

Average speed during a given interval. Measurement of magnitude and direction of velocity at a given moment.

2. Changes of position in one-dimensional motion can be analysed by means of a two-dimensional "position-time graph" drawn on squared paper. The gradient as measure of the speed at a given moment. Two-dimensional motion demands a tri-dimensional position-time graph or model; *e.g.* the position-time model for uniform motion of a point in a circle would be a uniform spiral round the time-line. An alternative method is to draw separate position-time graphs for the co-ordinates of the moving point, *i.e.* of its "projections" on two rectangular axes. These graphs may be regarded as the "plan" and "elevation" of the position-time model; their gradients at a given point measure the "rectangular components" of the velocity of the moving object. For tri-dimensional motion we must use three position-time graphs and no model is possible.

3. In studying one-dimensional motion it is also useful to use a graph in which the distance traversed during any given interval is represented by an area resting on a corresponding segment of the base-line. The ordinates of the resulting graph evidently represent the speed of the movement at the corresponding moments; for this reason it is called a "speed-time graph." The relation between the height of an ordinate and the area it marks off is the same as the relation between the "speed-law" and the "distance-law" of the motion. Note the specially simple and important case when the speed-law is $v=u+at$ and the distance-law is, therefore, $s=ut+\frac{1}{2}at^2$. Deduction of the law $v^2-u^2=2as$.

The "speed-time graph" of two-dimensional motion would, like the position-graph, be a tri-dimensional model. It is more convenient to dispense with a time-base and represent velocities at equal intervals by vectors drawn from a single point. The curve marked out by the distal ends of the vectors is called the *hodograph* of the motion. The vector joining the ends of any two of the representative vectors measures, in direction and magnitude, the change of velocity during the corresponding interval. Note that the hodograph of motion in a circle of any radius with uniform speed v is a circle of radius v; that if the speed increases uniformly the hodograph is a plane spiral, etc.

4. Angular motion. Measurement of angular displacement and velocity. Application of graphs and formulae for linear motion to angular displacement of a rotating wheel.

5. Uniform motion of a point in a circle as a simple instance of periodic motion. Note that the position-time graph of its projections on two diameters at right angles are respectively a sine-curve and a cosine-curve. Note also that if points travel in the same time round circles of different radii, the speeds of their projections at corresponding points of their movement are always proportional to the respective radii and, at the moment when the projections pass through the centres, are equal to the speeds of the points in the circles: that is, to $2\pi a/T$, where a is the radius of the circle and T the "period" of the motion.

6. Vibratory and pendulum motion as further instances of periodic movement. The period of a vibrating spring or a pendulum is found to be the same for all moderate amplitudes; *i.e.* the *average* speed during the swing is proportional to the amplitude a. This suggests that the speeds at *all* corresponding moments of the swing are proportional to the amplitude, as in the movements considered in § 5. This suggestion is confirmed by experiments showing that the position-time graph is a harmonic (*i.e.* sine- or cosine-) curve. It follows that the speed of the vibrating point when in its mean position is $2\pi a/T$, where a is the amplitude of swing (see B. § 1).

7. Projectile motion. Simple experiments indicate that when bodies are projected horizontally (*e.g.* from a shelf) with different speeds, (i) the time of fall to the ground depends only on the height of the point of projection, (ii) the "horizontal range" depends upon the speed of projection, (iii) the trajectory is parabolic. These facts suggest that the vertical and horizontal projections of the moving particle (see § 2) obey the distance-laws $y=\frac{1}{2}gt^2$ and $x=ut$ respectively, where g is a constant and u is the initial (horizontal) speed. It would follow that the speed-law for the vertical projection is $v=gt$ and that the speed of the horizontal projection is constant. Experiments on horizontal projection from different heights (*e.g.* from windows of a building) confirm both hypotheses and indicate that g is approximately 32 when distances are measured in feet. More direct confirmation is supplied by Goodwill's method of photographing an obliquely projected ball under intermittent illumination. The hodograph of the motion is drawn, and shows that, during both ascent and descent, the change of velocity in equal intervals is vertically downward and constant.

8. Arithmetical and graphical exercises on the foregoing.

B. *Dynamical Action.*

1. The changes of velocity produced by the collision of swinging balls studied (by means of Goodwill's "Vector Balance") as simple instances of dynamical action. It is found (i) that the changes in velocity are oppositely directed along the same straight line, and (ii) that the ratio of their magnitudes is always the inverse ratio of the numbers which measure the weights of the balls. That is, if the number which measures the weight of the ball is multiplied by the magnitude of the velocity-change, the product is the same for each ball. This product may be taken, therefore, as the measure of the "action" of ball A upon ball B and of the "reaction" of B upon A. "Action and reaction are equal and opposite."

2. Since weight is a vertically directed phenomenon and the collisions are horizontal, the weights of the balls cannot, as such, be involved in the results of § 1. There is no reason to believe that those results would be any different even if the collisions were brought about in a place where (as in Jules Verne's *Journey to the Moon*) things have no weight at all. We must suppose then, that every body has a property, called its *mass*, which belongs to it wherever it may be, and is a constant factor in determining the action it produces upon another body.

It must be noted that two bodies which have equal weight when weighed on a delicate balance side by side with one another are found not to have the same weight when one is a few feet nearer the ground than the other. We must say, then, that the number we take as the measure of a body's mass is the number which measures its weight when it is weighed *side by side* with standard pound-weights or gram-weights.

This is equivalent to saying that what is called a "standard weight" is really a "standard mass."

If m is the mass of a body and v its velocity, the product mV is called its *momentum*. If, by reason of the action of some other body, its velocity suffers a change, v, the change of momentum, mv, is the measure of the action.

The principle of the Conservation of Momentum.

3. In a collision the change of momentum which measures the action of one body upon the other is effected almost instantaneously. Contrast with this a case in which two bodies continuously change one another's velocities during a sensible period (*e.g.* two floating magnets attracting one another). In such cases we can determine not only the amount of the action between the bodies in a given time but the rate at which the action is being produced at a given moment. This rate—that is, the rate at which the momentum of each of the bodies is changing—is called the *force* between them at the given moment. The simplest case will be when the velocity of each body changes uniformly. We may then say that the force between them is the change of momentum produced in each of them per second. If this change is ma, the force, F, is said to be *ma poundals* (or *dynes*).

4. A falling body or a projectile is a case of this kind, for its velocity changes by g ft. (or *cms.*) per second downwards every second. Assuming that this change is due to the action of the earth, we must say, then, that the force between the earth and the body is mg poundals (or dynes), m being its mass. But we are accustomed to think of the action of the earth upon the body as the body's weight and to measure it as m lbs.-weight (or grams-weight). We conclude, on the one hand, that what we have hitherto measured as a "tension" or "thrust" (a "pull" or "push") of m lb.-wt. (or grms.-wt.) may be alternatively regarded as a "force" of mg poundals (or dynes), and, on the other hand, that a "force," made manifest as a change of momentum at the rate of ma units per second and called, therefore, a force of ma poundals (or dynes), may alternatively be spoken of as a force of ma/g lbs.-wt. (or grms.-wt.). Force-measurement in poundals (dynes) is called "absolute," in lb.-wt. (grams-wt.) "gravitational."

5. If the weight of a body (as determined by a sensitive balance) varies in different positions, the value of g should also vary. This is found to be the case; a pendulum swings more rapidly at a low than at a high altitude (see § 6). Newton surmised that the action of the earth on a mass would vary inversely as the square of its distance from the earth's centre (Law of Gravitation). He confirmed his hypothesis by calculating the distance the moon falls towards the earth in one second. If g' is the value of the moon's acceleration towards the earth, this distance is $\frac{1}{2}g'$. Let the moon's distance be D, the earth's radius R, and the time of the moon's revolution T, then it follows (Euclid, III. 35) that $g' = 4\pi^2 D/T^2$ (approx.). Substituting values (in ft. and secs.) it is found that $g'/g = (R/D)^2$.

6. Energy. A suspended ball is made to swing through a constant vertical distance along various curves, and to collide directly with a stationary suspended ball. The velocity immediately before impact is thus shown to depend entirely upon the vertical distance fallen. It must, then, be given by $v^2 = 2gh$ (see § 3). Connexion of result with Principle of Work (see Second Year, A. § 9). Kinetic energy. Potential energy. Foot-pounds, foot-poundals, kilogram-metres, ergs. Apparent loss of energy in collisions (to be considered in connexion with the study of the mechanical equivalent of heat in the Physics course).

When a pendulum of length l swings with amplitude a, its bob (Euclid,

III. 35) descends a distance $h=a^2/2l$. Hence its velocity in the mean position is $a\sqrt{(g/l)}$, and the time of complete swing (see A. § 5) is $2\pi\sqrt{(l/g)}$.

7. Exercises and problems on the foregoing.

C. *Statical Problems.*

1. Couples. Representation of the "torque" of a couple by a vector. Composition and resolution of couples. Work done by a couple.

2. Equivalence of a thrust or tension at a given point of a body to an equal thrust or tension at another given point, together with a determinable couple. Reduction of the whole of the thrusts and tensions acting on a body to a single thrust or tension and a single couple. Application to general problem of the equilibrium of a body. Simple exercises and experiments in illustration.

Fourth Year.

The work of this year consists mainly in a further development of the dynamical principles established in the preceding year, including an important extension to the simpler problems of "rigid dynamics." It will be seen, further, that the course now returns to a close association with the course in Physics.

1. Revision of work of Second and Third Years; straightforward problems on motion and equilibrium to give firm grasp of principles.

Power (rate of doing work); the horse-power and the kilo-watt; dynamometers. Work of engines in road, rail and water traffic. Economy of power.

Simple theory of the aeroplane.

2. Further consideration of the hodograph. If a point be supposed to trace out the hodograph in correspondence with the movement of the body under consideration, its velocity at any moment gives the rate of change of the velocity (*i.e.* the acceleration) of the body. Applied to the case of motion of a point with uniform speed v in a circle of radius r, this principle shows that the point has an acceleration, always directed to the centre of the circle, and of magnitude $v\times v/r=v^2/r$. Problems on circular motion, including the following (rough) demonstration of Kepler's Third Law. By Newton's Law of Gravitation the force on a planet of mass m is kMm/r, where M is the sun's mass, r the radius of the planet's orbit (assumed to be a circle with the sun at its centre), and k a constant. If the planet's speed is v, this force must also be mv^2/r, whence $v^2r=kM=$ constant. But if T is the planet's periodic time, $v=2\pi r/T$. Hence r^3/T^2 is constant for all planets of the system. Verification in the case of the solar system.

3. Harmonic motion. The results of § 2 used to show that when a point is moving uniformly with velocity v in a circle of radius a its projection upon a diameter has an acceleration $=-p^2 \,.\, x$, where $p=v/a$. Hence in this type of motion ("harmonic motion") the periodic time $T=2\pi/p$. Application to prove that wherever Hooke's Law holds good the free motion of the deformed body or system must be harmonic. Experiments in verification. Special application to the case of the simple pendulum (see Third Year, B. § 6).

4. Harmonic waves. The formula $y=a\sin\dfrac{2\pi}{\lambda}(x\pm vt)$ as descriptive of progressive waves. Stationary waves produced by the coalescence of trains of equal progressive waves travelling in opposite directions. Construction and interpretation of formulae describing such waves. Wave-motion as a mode of transmission of energy.

5. The dynamics of rotation. The energy of a body rotating about a fixed axis can be expressed as $\omega^2\Sigma(mr^2)$, where ω is the angular velocity of rotation. The product-sum $\Sigma(mr^2)$ is called the *moment of inertia*, and depends only on the form of the body and the position of the axis. It may be represented by the symbol I. If a couple, of torque T, acts on the body while it turns through an angle θ, it will do work $=T\theta$. This will be expressed in a change of the kinetic energy, so that $T\theta=\frac{1}{2}(\omega_2^2-\omega_1^2)I$. But (by analogy with the formulae for linear motion) $\omega_2-\omega_1=2a\theta$, where a is the angular acceleration. Hence the formula $T=Ia$ (analogous to the formula $F=ma$). $I\omega$ is called the *angular momentum* or *moment of momentum*; Ia is, therefore, the rate of change of the angular momentum.

Application of these principles to phenomena of cycling, spinning tops, the gyroscope, etc. Calculation of moment of inertia in simple cases. Motion of a rod struck at a given point.

Harmonic motion of a compound pendulum and of a horizontally suspended magnet. Inversion of a compound pendulum; "centre of percussion."

The case of a particle moving under the action of a force directed to a fixed point, *e.g.* a planet moving round the sun. Here, since there is no torque, the angular acceleration is zero, *i.e.* $mr^2 . \omega=$ a constant. Since $r^2 . \omega=r . r\omega=$ twice the rate at which the radius vector is sweeping out area, we have Kepler's Second Law that the radius vector of a planet sweeps out equal areas in equal times.

Note.—Where the subject is continued in an Advanced course the work should include a demonstration that Kepler's First Law follows from Newton's Law of Gravitation. Clerk Maxwell (*Matter and Motion*) has given a simple proof, but it involves a knowledge of the geometry of the ellipse which cannot be assumed to lie within the scope of the non-specialist student. The Advanced Course should also include a development of the mathematical physics of the Fourth Year, covering the "properties of matter," an elementary treatment of attractions and the simpler dynamics of wave-motion, with the more direct applications in acoustics, light and electricity. T. P. NUNN.

APPENDIX C.

Information from answers to questionnaire on the teaching of Mechanics in schools, issued November, 1916.

The questionnaire was issued to English, Scotch, and Welsh schools (including some girls' schools) in which it might be presumed that mechanics was taught. The most important information derived from the answers received, which numbered rather under two hundred, is as follows.

Mechanics is taught in 85 per cent. of these schools.

It is taken both as part of mathematics and as part of science in 44 per cent. of these, as part of science only, in 32 per cent., and as part of mathematics only, in 24 per cent.

The subject is usually begun at about age fourteen when taught as part of mathematics, but earlier when taught as part of science.

The number of hours a week given to mechanics varies considerably, but few give less than two periods a week.

The subject is taught experimentally in 88 per cent., and experiments are performed by the boys themselves in 83 per cent.

Hydrostatics is included with mechanics in the great majority of those schools which teach mechanics as part of science, but in most

of those in which mechanics is part of mathematics, hydrostatics is taken separately.

Trigonometry is not usually begun before mechanics in schools where mechanics is part of science, but where mechanics is part of mathematics, some trigonometry has usually been done first.

The calculus is used in connection with mechanics in a large number of schools. Most, however, begin mechanics before the calculus.

Statics is taken up before kinetics in the great majority of schools.

In statics the number of those who begin with the triangle or parallelogram law is rather in excess of those who begin with the law of the lever.

In kinetics the great majority begin with linear motion.

The unit of force used in kinetics is more frequently the poundal than the pound weight.

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